International Journal of Mathematics and Physical Sciences Research ISSN 2348-5736 (Online)
Vol. 10, Issue 1, pp: (52-58), Month: April 2022 - September 2022, Available at: www.researchpublish.com

A Study of Some Fractional Differential Problems

Chii-Huei Yu

School of Mathematics and Statistics, Zhaoqing University, Guangdong, China

DOI: https://doi.org/10.5281/zenodo.7064096

Published Date: 09-September-2022

Abstract: **This paper studies the fractional differential problems of two types of fractional analytic functions based on Jumarie's modified Riemann-Liouville (R-L) fractional derivative. Any order fractional derivatives of these two types of fractional analytic functions can be obtained by using fractional Euler's formula, fractional DeMoivre's formula, and fractional Fourier series method. A new multiplication of fractional analytic functions plays an important role in this paper. In fact, our results are generalizations of the results of classical calculus. On the other hand, two examples are provided to illustrate our results.**

Keywords: **fractional differential problems, fractional analytic functions, Jumarie's modified R-L fractional derivative, fractional Euler's formula, fractional DeMoivre's formula, fractional Fourier series, new multiplication.**

I. INTRODUCTION

Fractional calculus has attracted the attention of scientists and engineers since a long time ago, leading to the development of many applications. Since the 1990s, fractional calculus has been rediscovered and applied in more and more fields, such as physics, control engineering, mechanics, dynamics, modeling, signal processing, biology, economics, electrical engineering, and so on [1-10].

However, the definition of fractional derivative is not unique, there are many useful definitions, including Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, Jumarie's modified R-L fractional derivative [11-16]. Jumarie revised the definition of R-L fractional derivative with a new formula, and we obtained that the modified fractional derivative of a constant function is zero. Therefore, by using this definition, it is easier to link fractional calculus with traditional calculus.

In this paper, based on Jumarie type of R-L fractional derivative, we discuss the fractional differential problems of the following two types of α-fractional analytic functions:

$$Ln_{\alpha}\left(1+r^{2}+2r\cdot cos_{\alpha}(\theta^{\alpha})\right), \tag{1}$$

and

$$arctan[(r\cdot sin_{\alpha}(\theta^{\alpha})\otimes[1+r\ \cdot cos_{\alpha}(\theta^{\alpha})]^{\otimes -1})]. \tag{2}$$

Where $0<\alpha\leq 1$, and r is a real number, $|r|<1$. We can use fractional Euler's formula, fractional DeMoivre's formula, and complex fractional analytic function method to obtain the arbitrary order fractional derivatives of the above fractional analytic functions. A new multiplication of fractional analytic functions plays an important role in this article. In fact, the results we obtained are generalizations of traditional calculus results. On the other hand, some examples are given to illustrate our results.

II. DEFINITIONS AND PROPERTIES

First, we introduce fractional calculus and some important properties used in this paper.

Definition 2.1 ([17])**:** If $0<\alpha\leq 1$, and θ_0 is a real number. The Jumarie's modified R-L α-fractional derivative is defined by

$$\left({}_{\theta_0}D^{\alpha}_{\theta}\right)[f(\theta)]=\frac{1}{\Gamma(1-\alpha)}\frac{d}{d\theta}\int_{\theta_0}^{\theta}\frac{f(x)-f(\theta_0)}{(\theta-x)^{\alpha}}dx, \tag{3}$$

where $\Gamma(\ )$ is the gamma function. In addition, we define $\left({}_{\theta_0}I_\theta^\alpha\right)^n[f(\theta)] = \left({}_{\theta_0}I_\theta^\alpha\right)\left({}_{\theta_0}I_\theta^\alpha\right)\cdots\left({}_{\theta_0}I_\theta^\alpha\right)[f(\theta)]$, and it is called the n-th order α-fractional derivative of $f(\theta)$, where n is a positive integer.

Proposition 2.2 ([18])**:** *If* $\alpha, \beta, \theta_0, C$ *are real numbers and* $\beta \geq \alpha > 0$, *then*

$$\left({}_{\theta_0}D_\theta^\alpha\right)\left[(\theta-\theta_0)^\beta\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(\theta-\theta_0)^{\beta-\alpha}, \tag{4}$$

and

$$\left({}_{\theta_0}D_\theta^\alpha\right)[C] = 0. \tag{5}$$

In the following, we introduce the definition of fractional analytic function.

Definition 2.3 ([19])**:** If θ, θ_0, and a_k are real numbers for all k, $\theta_0 \in (a,b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, that is, $f_\alpha(\theta^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(\theta-\theta_0)^{k\alpha}$ on some open interval containing θ_0, then we say that $f_\alpha(\theta^\alpha)$ is α-fractional analytic at θ_0. In addition, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

Definition 2.4 ([20])**:** If $0 < \alpha \leq 1$, and θ_0 is a real number. Let $f_\alpha(\theta^\alpha)$ and $g_\alpha(\theta^\alpha)$ be two α-fractional analytic functions defined on an interval containing θ_0 ,

$$f_\alpha(\theta^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(\theta-\theta_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(\theta-\theta_0)^\alpha\right)^{\otimes k}, \tag{6}$$

$$g_\alpha(\theta^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(\theta-\theta_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(\theta-\theta_0)^\alpha\right)^{\otimes k}. \tag{7}$$

Then

$$\begin{aligned} & f_\alpha(\theta^\alpha) \otimes g_\alpha(\theta^\alpha) \\ & = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(\theta-\theta_0)^{k\alpha} \otimes \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(\theta-\theta_0)^{k\alpha} \\ & = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)(\theta-\theta_0)^{k\alpha}. \end{aligned} \tag{8}$$

Equivalently,

$$\begin{aligned} & f_\alpha(\theta^\alpha) \otimes g_\alpha(\theta^\alpha) \\ & = \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(\theta-\theta_0)^\alpha\right)^{\otimes k} \otimes \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(\theta-\theta_0)^\alpha\right)^{\otimes k} \\ & = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(\theta-\theta_0)^\alpha\right)^{\otimes k}. \end{aligned} \tag{9}$$

Definition 2.5 ([21])**:** Suppose that $0 < \alpha \leq 1$, and $f_\alpha(\theta^\alpha)$, $g_\alpha(\theta^\alpha)$ are α-fractional analytic at $\theta = \theta_0$,

$$f_\alpha(\theta^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(\theta-\theta_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(\theta-\theta_0)^\alpha\right)^{\otimes k}, \tag{10}$$

$$g_\alpha(\theta^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(\theta-\theta_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(\theta-\theta_0)^\alpha\right)^{\otimes k}. \tag{11}$$

The compositions of $f_\alpha(\theta^\alpha)$ and $g_\alpha(\theta^\alpha)$ are defined by

$$(f_\alpha \circ g_\alpha)(\theta^\alpha) = f_\alpha\left(g_\alpha(\theta^\alpha)\right) = \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(g_\alpha(\theta^\alpha)\right)^{\otimes k}, \tag{12}$$

and

$$(g_\alpha \circ f_\alpha)(\theta^\alpha) = g_\alpha\left(f_\alpha(\theta^\alpha)\right) = \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(f_\alpha(\theta^\alpha)\right)^{\otimes k}. \tag{13}$$

Definition 2.6 ([21])**:** Let $0 < \alpha \leq 1$. If $f_\alpha(\theta^\alpha)$, $g_\alpha(\theta^\alpha)$ are two α-fractional analytic functions satisfies

$$(f_\alpha \circ g_\alpha)(\theta^\alpha) = (g_\alpha \circ f_\alpha)(\theta^\alpha) = \frac{1}{\Gamma(\alpha+1)}\theta^\alpha. \tag{14}$$

Then $f_\alpha(\theta^\alpha)$, $g_\alpha(\theta^\alpha)$ are called inverse functions of each other.

Definition 2.7 ([21])**:** If $0 < \alpha \le 1$, and θ is any real number. The α-fractional exponential function is defined by

$$E_\alpha(\theta^\alpha) = \sum_{k=0}^{\infty}\frac{\theta^{k\alpha}}{\Gamma(k\alpha+1)} = \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{1}{\Gamma(\alpha+1)}\theta^\alpha\right)^{\otimes k}. \tag{15}$$

And $Ln_\alpha(\theta^\alpha)$ is the inverse function of $E_\alpha(\theta^\alpha)$. In addition, the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(\theta^\alpha) = \sum_{k=0}^{\infty}\frac{(-1)^k\theta^{2k\alpha}}{\Gamma(2k\alpha+1)} = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\left(\frac{1}{\Gamma(\alpha+1)}\theta^\alpha\right)^{\otimes 2k}, \tag{16}$$

and

$$sin_\alpha(\theta^\alpha) = \sum_{k=0}^{\infty}\frac{(-1)^k\theta^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)} = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\left(\frac{1}{\Gamma(\alpha+1)}\theta^\alpha\right)^{\otimes(2k+1)}. \tag{17}$$

Definition 2.8 ([22])**:** If $0 < \alpha \le 1$, and r is any real number. Let $f_\alpha(\theta^\alpha)$ be a α-fractional analytic function. Then the α-fractional analytic function $f_\alpha(\theta^\alpha)^{\otimes r}$ is defined by

$$f_\alpha(\theta^\alpha)^{\otimes r} = E_\alpha\left(r \cdot Ln_\alpha\left(f_\alpha(\theta^\alpha)\right)\right). \tag{18}$$

Definition 2.9 ([23])**:** Suppose that $0 < \alpha \le 1$, $i = \sqrt{-1}$, and $f_\alpha(\theta^\alpha), g_\alpha(\theta^\alpha), p_\alpha(\theta^\alpha), q_\alpha(\theta^\alpha)$ are α-fractional real analytic at $\theta = \theta_0$. Let $z_\alpha(\theta^\alpha) = f_\alpha(\theta^\alpha) + ig_\alpha(\theta^\alpha)$ and $w_\alpha(\theta^\alpha) = p_\alpha(\theta^\alpha) + iq_\alpha(\theta^\alpha)$ be complex analytic at $\theta = \theta_0$. Define

$$\begin{aligned} & z_\alpha(\theta^\alpha)\otimes w_\alpha(\theta^\alpha) \\ &= \left(f_\alpha(\theta^\alpha) + i\, g_\alpha(\theta^\alpha)\right)\otimes\left(p_\alpha(\theta^\alpha) + i\, q_\alpha(\theta^\alpha)\right) \\ &= [f_\alpha(\theta^\alpha)\otimes p_\alpha(\theta^\alpha) - g_\alpha(\theta^\alpha)\otimes q_\alpha(\theta^\alpha)] + i[f_\alpha(\theta^\alpha)\otimes q_\alpha(\theta^\alpha) + g_\alpha(\theta^\alpha)\otimes p_\alpha(\theta^\alpha)]. \end{aligned} \tag{19}$$

Moreover, we define

$$|\, z_\alpha(\theta^\alpha)|_\otimes = ([f_\alpha(\theta^\alpha)]^{\otimes 2} + [g_\alpha(\theta^\alpha)]^{\otimes 2})^{\otimes\frac{1}{2}}. \tag{20}$$

Proposition 2.10 (fractional Euler's formula) ([24])**:** *Let* $0 < \alpha \le 1$, θ *be a real number, then*

$$E_\alpha(i\theta^\alpha) = cos_\alpha(\theta^\alpha) + isin_\alpha(\theta^\alpha)\,. \tag{21}$$

Proposition 2.11 (fractional DeMoivre's formula) ([24])**:** *Let* $0 < \alpha \le 1$, n *be any positive integer, then*

$$\left(cos_\alpha(\theta^\alpha) + isin_\alpha(\theta^\alpha)\right)^{\otimes n} = cos_\alpha(n\theta^\alpha) + isin_\alpha(n\theta^\alpha)\,. \tag{22}$$

Theorem 2.12 (chain rule for fractional derivative) ([24])**:** Let $0 < \alpha \le 1$, $g_\alpha(\theta^\alpha)$ be α-fractional analytic at $\theta = \theta_0$ and $f_\alpha(\theta^\alpha)$ be α-fractional analytic at $\theta = g_\alpha(\theta_0^\alpha)$. Then

$$\left({}_{\theta_0}D_\theta^\alpha\right)[\,(f_\alpha \circ g_\alpha)(\theta^\alpha)] = \left({}_{g_\alpha(\theta_0^\alpha)}D_\theta^\alpha\right)[\,f_\alpha(\theta^\alpha)](g_\alpha(\theta^\alpha))\otimes\left({}_{\theta_0}D_\theta^\alpha\right)[g_\alpha(\theta^\alpha)]. \tag{23}$$

Definition 2.13 (fractional Fourier series) ([23])**:** If $0 < \alpha \le 1$, and $f_\alpha(\theta^\alpha)$ is α-fractional analytic at $\theta = 0$ with the same period T_α of $E_\alpha(i\theta^\alpha)$. Then the α-fractional Fourier series expansion of $f_\alpha(\theta^\alpha)$ is

$$\frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k cos_\alpha(k\theta^\alpha) + b_k sin_\alpha(k\theta^\alpha), \tag{24}$$

where
$$\begin{cases} a_0 = \frac{2}{T_\alpha}\left({}_0I_{T_\alpha}^\alpha\right)[f_\alpha(\theta^\alpha)], \\ a_k = \frac{2}{T_\alpha}\left({}_0I_{T_\alpha}^\alpha\right)[f_\alpha(\theta^\alpha)\otimes cos_\alpha(k\theta^\alpha)], \\ b_k = \frac{2}{T_\alpha}\left({}_0I_{T_\alpha}^\alpha\right)[f_\alpha(\theta^\alpha)\otimes sin_\alpha(k\theta^\alpha)], \end{cases} \tag{25}$$

for all positive integers k.

In the following, the inverse fractional trigonometric functions are introduced.

Definition 2.14 ([25])**:** Let $0 < \alpha \leq 1$. Then $arcsin_\alpha(\theta^\alpha)$ is the inverse function of $sin_\alpha(\theta^\alpha)$, and it is called inverse α-fractional sine function. $arccos_\alpha(\theta^\alpha)$ is the inverse function of $cos_\alpha(\theta^\alpha)$, and we say that it is the inverse α-fractional cosine function. On the other hand, $arctan_\alpha(\theta^\alpha)$ is the inverse function of $tan_\alpha(\theta^\alpha)$, and it is called the inverse α-fractional tangent function. $arccotan_\alpha(\theta^\alpha)$ is the inverse function of $cot_\alpha(\theta^\alpha)$, and we say that it is the inverse α-fractional cotangent function. $arcsec_\alpha(\theta^\alpha)$ is the inverse function of $sec_\alpha(\theta^\alpha)$, and it is the inverse α-fractional secant function. $arccsc_\alpha(\theta^\alpha)$ is the inverse function of $csc_\alpha(\theta^\alpha)$, and it is called the inverse α-fractional cosecant function.

III. RESULTS AND EXAMPLES

To obtain the major results in this paper, the following lemmas are needed.

Lemma 3.1: *If* $0 < \alpha \leq 1$, *and* n *is any positive integer, then*

$$\left({}_0D_\theta^\alpha\right)^n[sin_\alpha(\theta^\alpha)] = sin_\alpha\left(\theta^\alpha + n\cdot\tfrac{T_\alpha}{4}\right), \tag{26}$$

and

$$\left({}_0D_\theta^\alpha\right)^n[cos_\alpha(\theta^\alpha)] = cos_\alpha\left(\theta^\alpha + n\cdot\tfrac{T_\alpha}{4}\right). \tag{27}$$

Proof If $n = 1$, then $\left({}_0D_\theta^\alpha\right)[sin_\alpha(\theta^\alpha)] = cos_\alpha(\theta^\alpha) = sin_\alpha\left(\theta^\alpha + \tfrac{T_\alpha}{4}\right)$. Suppose that $n = m$ holds. That is,

$$\left({}_0D_\theta^\alpha\right)^m[sin_\alpha(\theta^\alpha)] = sin_\alpha\left(\theta^\alpha + m\cdot\tfrac{T_\alpha}{4}\right).$$

Then if $n = m + 1$,

$$\begin{aligned}&\left({}_0D_\theta^\alpha\right)^{m+1}[sin_\alpha(\theta^\alpha)]\\ &= \left({}_0D_\theta^\alpha\right)\left[\left({}_0D_\theta^\alpha\right)^m[sin_\alpha(\theta^\alpha)]\right]\\ &= \left({}_0D_\theta^\alpha\right)\left[sin_\alpha\left(\theta^\alpha + m\cdot\tfrac{T_\alpha}{4}\right)\right]\\ &= cos_\alpha\left(\theta^\alpha + m\cdot\tfrac{T_\alpha}{4}\right)\\ &= sin_\alpha\left(\theta^\alpha + (m+1)\cdot\tfrac{T_\alpha}{4}\right).\end{aligned} \tag{28}$$

By induction, Eq. (26) holds. Similarly, we can easily show that Eq. (27) holds. Q.e.d.

Lemma 3.2: *Let* $0 < \alpha \leq 1$ *and if* $z_\alpha(\theta^\alpha)$ *is a complex* α*-fractional analytic function such that* $|z_\alpha(\theta^\alpha)|_\otimes < 1$. *Then*

$$Ln_\alpha\big(1 + z_\alpha(\theta^\alpha)\big) = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}[z_\alpha(\theta^\alpha)]^{\otimes k}\ . \tag{29}$$

Proof By chain rule for fractional derivative, we have

$$\left({}_0D_\theta^\alpha\right)\left[\,Ln_\alpha\big(1 + z_\alpha(\theta^\alpha)\big)\right] = [1 + z_\alpha(\theta^\alpha)]^{\otimes -1}\otimes\left({}_0D_\theta^\alpha\right)[z_\alpha(\theta^\alpha)]. \tag{30}$$

And

$$\left({}_0D_\theta^\alpha\right)\left[\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}[z_\alpha(\theta^\alpha)]^{\otimes k}\right] = [1 + z_\alpha(\theta^\alpha)]^{\otimes -1}\otimes\left({}_0D_\theta^\alpha\right)[z_\alpha(\theta^\alpha)]. \tag{31}$$

Therefore,

$$Ln_\alpha\big(1 + z_\alpha(\theta^\alpha)\big) = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}[z_\alpha(\theta^\alpha)]^{\otimes k}\ .$$ Q.e.d.

Lemma 3.3: *If* $0 < \alpha \leq 1$, r *is a real number and* $|r| < 1$. *Then*

$$Ln_\alpha\big(1 + r^2 + 2r\cdot cos_\alpha(\theta^\alpha)\big) = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}r^k cos_\alpha(k\theta^\alpha). \tag{32}$$

And

$$arctan[(r\cdot sin_\alpha(\theta^\alpha)\otimes[1 + r\ \cdot cos_\alpha(\theta^\alpha)]^{\otimes -1})] = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}r^k sin_\alpha(k\theta^\alpha). \tag{33}$$

Proof Let $z_\alpha(\theta^\alpha) = r \cdot E_\alpha(i\theta^\alpha)$, then by fractional Euler's formula, $z_\alpha(\theta^\alpha) = r \cdot cos_\alpha(\theta^\alpha) + ir \cdot sin_\alpha(\theta^\alpha)$. Thus, by Lemma 3.2, we obtain

$$Ln_\alpha\left(1 + r \cdot E_\alpha(i\theta^\alpha)\right) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} [r \cdot E_\alpha(i\theta^\alpha)]^{\otimes k}. \quad (34)$$

Using fractional DeMoivre's formula yields

$$Ln_\alpha\left(1 + r \cdot cos_\alpha(\theta^\alpha) + ir \cdot sin_\alpha(\theta^\alpha)\right) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} r^k E_\alpha(ik\theta^\alpha). \quad (35)$$

Hence,

$$Ln_\alpha\left(\left[\left(1 + r \cdot cos_\alpha(\theta^\alpha)\right)^{\otimes 2} + \left(r \cdot sin_\alpha(\theta^\alpha)\right)^{\otimes 2}\right]\right)$$
$$+Ln_\alpha\left(\left[\left(1 + r \cdot cos_\alpha(\theta^\alpha)\right)^{\otimes 2} + \left(r \cdot sin_\alpha(\theta^\alpha)\right)^{\otimes 2}\right]^{\otimes -1} \otimes [1 + r \cdot cos_\alpha(\theta^\alpha) + ir \cdot sin_\alpha(\theta^\alpha)]\right)$$
$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} r^k \left(cos_\alpha(k\theta^\alpha) + i sin_\alpha(k\theta^\alpha)\right). \quad (36)$$

Therefore,

$$Ln_\alpha\left(1 + r^2 + 2r \cdot cos_\alpha(\theta^\alpha)\right)$$
$$+Ln_\alpha([1 + r^2 + 2r \cdot cos_\alpha(\theta^\alpha)]^{\otimes -1} \otimes [1 + r \cdot cos_\alpha(\theta^\alpha) + ir \cdot sin_\alpha(\theta^\alpha)])$$
$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} r^k \left(cos_\alpha(k\theta^\alpha) + i sin_\alpha(k\theta^\alpha)\right). \quad (37)$$

So,

$$Ln_\alpha\left(1 + r^2 + 2r \cdot cos_\alpha(\theta^\alpha)\right)$$
$$+Ln_\alpha\left(E_\alpha(i arctan[(r \cdot sin_\alpha(\theta^\alpha) \otimes [1 + r \cdot cos_\alpha(\theta^\alpha)]^{\otimes -1})])\right)$$
$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} r^k cos_\alpha(k\theta^\alpha) + i \cdot \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} r^k sin_\alpha(k\theta^\alpha). \quad (38)$$

Thus,

$$Ln_\alpha\left(1 + r^2 + 2r \cdot cos_\alpha(\theta^\alpha)\right) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} r^k cos_\alpha(k\theta^\alpha).$$

And

$$arctan[(r \cdot sin_\alpha(\theta^\alpha) \otimes [1 + r \cdot cos_\alpha(\theta^\alpha)]^{\otimes -1})] = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} r^k sin_\alpha(k\theta^\alpha).$$

Q.e.d.

The following is the major result in this paper.

Theorem 3.4: *Let* $0 < \alpha \le 1$, *and* n *be any positive integer, then the* n*-th order* α*-fractional derivatives*

$$\left({}_0D_\theta^\alpha\right)^n \left[Ln_\alpha\left(1 + r^2 + 2r \cdot cos_\alpha(\theta^\alpha)\right)\right] = \sum_{k=1}^{\infty} (-1)^{k+1} r^k k^{n-1} cos_\alpha\left(k\theta^\alpha + n \cdot \frac{T_\alpha}{4}\right). \quad (39)$$

And

$$\left({}_0D_\theta^\alpha\right)^n \left[arctan[(r \cdot sin_\alpha(\theta^\alpha) \otimes [1 + r \cdot cos_\alpha(\theta^\alpha)]^{\otimes -1})]\right] = \sum_{k=1}^{\infty} (-1)^{k+1} r^k k^{n-1} sin_\alpha\left(k\theta^\alpha + n \cdot \frac{T_\alpha}{4}\right). \quad (40)$$

Proof By Lemmas 3.1 and 3.3, we have

$$\left({}_0D_\theta^\alpha\right)^n \left[Ln_\alpha\left(1 + r^2 + 2r \cdot cos_\alpha(\theta^\alpha)\right)\right]$$
$$= \left({}_0D_\theta^\alpha\right)^n \left[\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} r^k cos_\alpha(k\theta^\alpha)\right]$$
$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} r^k cos_\alpha(k\theta^\alpha) \left({}_0D_\theta^\alpha\right)^n [cos_\alpha(k\theta^\alpha)]$$

$$= \sum_{k=1}^{\infty}(-1)^{k+1} r^k k^{n-1} cos_\alpha\left(k\theta^\alpha + n \cdot \frac{T_\alpha}{4}\right).$$

And

$$\left({}_0D_\theta^\alpha\right)^n\left[arctan[(r \cdot sin_\alpha(\theta^\alpha) \otimes [1 + r \cdot cos_\alpha(\theta^\alpha)]^{\otimes -1})]\right]$$

$$= \left({}_0D_\theta^\alpha\right)^n\left[\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k} r^k sin_\alpha(k\theta^\alpha)\right]$$

$$= \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k} r^k sin_\alpha(k\theta^\alpha)\left({}_0D_\theta^\alpha\right)^n[sin_\alpha(k\theta^\alpha)]$$

$$= \sum_{k=1}^{\infty}(-1)^{k+1} r^k k^{n-1} sin_\alpha\left(k\theta^\alpha + n \cdot \frac{T_\alpha}{4}\right).$$

Q.e.d.

Finally, two examples are provided to illustrate our results.

Example 3.5: Assume that $0 < \alpha \leq 1$, and let $r = \frac{1}{3}$, and $n = 6$, then using Theorem 3.4 yields

$$\left({}_0D_\theta^\alpha\right)^6\left[Ln_\alpha\left(\frac{10}{9} + \frac{2}{3} \cdot cos_\alpha(\theta^\alpha)\right)\right].$$

$$= \sum_{k=1}^{\infty}(-1)^{k+1}\left(\frac{1}{3}\right)^k k^5 cos_\alpha\left(k\theta^\alpha + 6 \cdot \frac{T_\alpha}{4}\right)$$

$$= \sum_{k=1}^{\infty}(-1)^{k}\left(\frac{1}{3}\right)^k k^5 cos_\alpha(k\theta^\alpha). \qquad (41)$$

Example 3.6: Let $0 < \alpha \leq 1$, and if $r = \frac{3}{4}$ and $n = 8$, then by Theorem 3.4,

$$\left({}_0D_\theta^\alpha\right)^8\left[arctan\left[\left(\frac{3}{4} \cdot sin_\alpha(\theta^\alpha) \otimes \left[1 + \frac{3}{4} \cdot cos_\alpha(\theta^\alpha)\right]^{\otimes -1}\right)\right]\right]$$

$$= \sum_{k=1}^{\infty}(-1)^{k+1}\left(\frac{3}{4}\right)^k k^7 sin_\alpha\left(k\theta^\alpha + 8 \cdot \frac{T_\alpha}{4}\right)$$

$$= \sum_{k=1}^{\infty}(-1)^{k+1}\left(\frac{3}{4}\right)^k k^7 sin_\alpha(k\theta^\alpha). \qquad (42)$$

IV. CONCLUSION

In this paper, we study the fractional differential problems of two kinds of fractional analytic functions. We can obtain any order fractional derivatives of these two kinds of fractional analytic functions by using fractional Euler's formula, fractional DeMoivre's formula, and fractional Fourier series method. A new multiplication of fractional analytic functions plays an important role in this paper. In fact, the results obtained in this paper are the extension of the results of ordinary calculus. On the other hand, some examples are given to illustrate our results. In the future, we will continue to use the above methods mentioned above to solve the problems in applied mathematics and fractional differential equations.

REFERENCES

[1] V. E. Tarasov, Mathematical economics: application of fractional calculus, Mathematics, vol. 8, no. 5, 660, 2020.

[2] Mohd. Farman Ali, Manoj Sharma, Renu Jain, An application of fractional calculus in electrical engineering, Advanced Engineering Technology and Application, vol. 5, no. 2, pp, 41-45, 2016.

[3] J. T. Machado, Fractional Calculus: Application in Modeling and Control, Springer New York, 2013.

[4] F. Mainardi, Fractional calculus: some basic problems in continuum and statistical mechanics, Fractals and Fractional Calculus in Continuum Mechanics, A. Carpinteri and F. Mainardi, Eds., pp. 291-348, Springer, Wien, Germany, 1997.

[5] R. Magin, Fractional calculus in bioengineering, part 1, Critical Reviews in Biomedical Engineering, vol. 32, no,1. pp.1-104, 2004.

[6] A. Dzielinski, D. Sierociuk, Ultracapacitor modelling and control using discrete fractional order state-space model, Acta Montanistica Slovaca, vol. 13, no. 1, pp. 136-145, 2008.

[7] C. Cattani, H. M. Srivastava, and X. -J. Yang, (Eds.), Fractional Dynamics, Emerging Science Publishers (De Gruyter Open), Berlin and Warsaw, 2015.

[8] C. -H. Yu, A study on fractional RLC circuit, International Research Journal of Engineering and Technology, vol. 7, no. 8, pp. 3422-3425, 2020.

[9] C. -H. Yu, Study on fractional Newton's law of cooling, International Journal of Mechanical and Industrial Technology, vol. 9, no. 1, pp. 1-6, 2021.

[10] C. -H. Yu, A new insight into fractional logistic equation, International Journal of Engineering Research and Reviews, vol. 9, no. 2, pp.13-17, 2021,

[11] S. Das, Functional Fractional Calculus for System Identification and Control, 2nd ed., Springer-Verlag, Berlin, 2011.

[12] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, Calif, USA, 1999.

[13] K. S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley & Sons, New York, USA, 1993.

[14] K. B. Oldham and J. Spanier, The Fractional Calculus, Academic Press, Inc., 1974.

[15] S. Das, Functional Fractional Calculus, 2nd ed. Springer-Verlag, 2011.

[16] K. Diethelm, The Analysis of Fractional Differential Equations, Springer-Verlag, 2010.

[17] C. -H. Yu, Fractional derivative of arbitrary real power of fractional analytic function, International Journal of Novel Research in Engineering and Science, vol. 9, no. 1, pp. 9-13, 2022.

[18] U. Ghosh, S. Sengupta, S. Sarkar and S. Das, Analytic solution of linear fractional differential equation with Jumarie derivative in term of Mittag-Leffler function, American Journal of Mathematical Analysis, vol. 3, no. 2, pp. 32-38, 2015.

[19] C. -H. Yu, Study of fractional analytic functions and local fractional calculus, International Journal of Scientific Research in Science, Engineering and Technology, vol. 8, no. 5, pp. 39-46, 2021.

[20] C. -H. Yu, A study on arc length of nondifferentiable curves, Research Inventy: International Journal of Engineering and Science, vol. 12, no. 4, pp. 18-23, 2022.

[21] C. -H. Yu, Using trigonometric substitution method to solve some fractional integral problems, International Journal of Recent Research in Mathematics Computer Science and Information Technology, vol. 9, no. 1, pp. 10-15, 2022.

[22] C. -H. Yu, A study on fractional derivative of fractional power exponential function, American Journal of Engineering Research, vol. 11, no. 5, pp. 100-103, 2022.

[23] C. -H. Yu, Fractional Fourier series expansion of two types of fractional trigonometric functions, International Journal of Electrical and Electronics Research, vol. 10, no. 3, pp. 4-9, 2022.

[24] C. -H. Yu, Differential properties of fractional functions, International Journal of Novel Research in Interdisciplinary Studies, vol. 7, no. 5, pp. 1-14, 2020.

[25] C. -H. Yu, Study of inverse fractional trigonometric functions, International Journal of Innovative Research in Science, Engineering and Technology, vol. 11, no. 3, pp. 2019-2026, 2022.